Continuous Variables II

Note Title 10/4/2010

# Quantum information with continuous variables*

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Quantum information is a rapidly advancing area of interdisciplinary research. It may lead to real-world applications for communication and computation unavailable without the exploitation of quantum properties such as nonorthogonality or entanglement. We review the progress in quantum information based on continuous quantum variables, with emphasis on quantum optical implementations in terms of the quadrature amplitudes of the electromagnetic field.

Revs. of. Modern Physics (2005)

## Contents

$$\nabla \cdot E = 0 \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t}$$

$$H = \frac{1}{2}\int (E^2 + B^2)\, d^3x$$ انرژی

$$\vec{P} = \int \vec{E} \times \vec{B}\, d^3x$$ تکانه

$$A(x,t) = \sum_k a_k e^{i(\omega_k t - k\cdot x)} + a_k^{+} e^{-i(\omega_k t - k\cdot x)}$$

حل معادلات ماکسول

$$H = \sum_k \hbar\omega_k \left( a_k^{+} a_k + \frac{1}{2} \right)$$

$$\vec{P} = \sum_k \hbar\vec{k} \left( a_k^{*} a_k \right)$$

quantization: $[a_k, a_{k'}^{\dagger}] = \delta_{k,k'}$

عملگر فنا

عملگر خلق

$a_k |\Omega\rangle = 0$ $\qquad$ $|\Omega\rangle = |0\rangle \otimes |0\rangle \otimes \cdots\cdots |0\rangle \cdots$

خلاء

$$\hat{H} = \sum_k w_k \left( a_k^\dagger a_k + \frac{1}{2} \right)$$

$$\hat{\vec{P}} = \sum_k \vec{k} \left( a_k^\dagger a_k \right)$$

$|\Omega\rangle$ = Vaccum = No photon

$a_k^\dagger |\Omega\rangle = |$ a photon with momentum $\vec{k}\,\rangle$

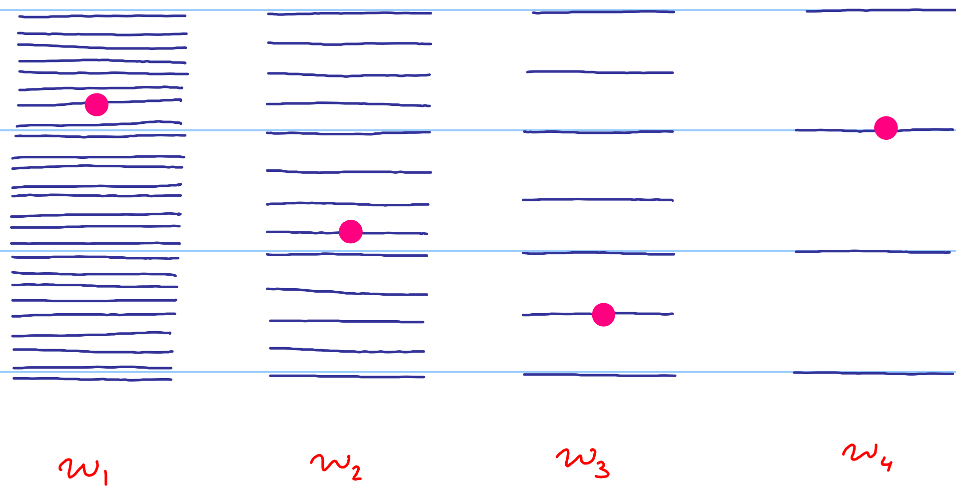
$a_k^\dagger a_p^\dagger |\Omega\rangle$ = two-photon state

$a_k^\dagger a_k$ = photon number operator

More precizely $\vec{a}_k \longrightarrow \vec{a}_{k,\lambda}$

Polarization

$|\, n_1 = 20,\ n_2 = 6,\ n_3 = 2,\ n_4 = 3,\ n_5 = 0,\ 0,\ 0,\ \ldots \rangle$




$$\hat{H}_k = \frac{1}{2}\left(\hat{p}_k^2 + \omega_k^2 \hat{x}_k^2\right),$$

with

$$\hat{a}_k = \frac{1}{\sqrt{2\hbar\omega_k}}\left(\omega_k \hat{x}_k + i\hat{p}_k\right),$$
$$\hat{a}_k^\dagger = \frac{1}{\sqrt{2\hbar\omega_k}}\left(\omega_k \hat{x}_k - i\hat{p}_k\right),$$

or, conversely,

$$\hat{x}_k = \sqrt{\frac{\hbar}{2\omega_k}}\left(\hat{a}_k + \hat{a}_k^\dagger\right),$$
$$\hat{p}_k = -i\sqrt{\frac{\hbar\omega_k}{2}}\left(\hat{a}_k - \hat{a}_k^\dagger\right).$$

$$\hat{E}_k(\mathbf{r},t) = E_0\left[\hat{a}_k\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + \hat{a}_k^\dagger\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right].$$

$$\hat{E}_k(\mathbf{r},t) = 2E_0\left[\hat{x}_k \cos(\omega_k t - \mathbf{k}\cdot\mathbf{r}) + \hat{p}_k \sin(\omega_k t - \mathbf{k}\cdot\mathbf{r})\right].$$

In phase      out of phase.

$\langle \psi | \hat{x}_k | \psi \rangle$, states of light

$\langle \psi | P_k | \psi \rangle$,

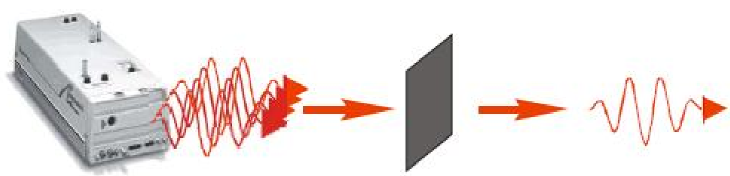
$\langle \psi | x_k x_\ell | \psi \rangle$, $|\psi\rangle \in$ Infinite dimensional space.

$|\psi\rangle = |n\rangle$ a one-mode state

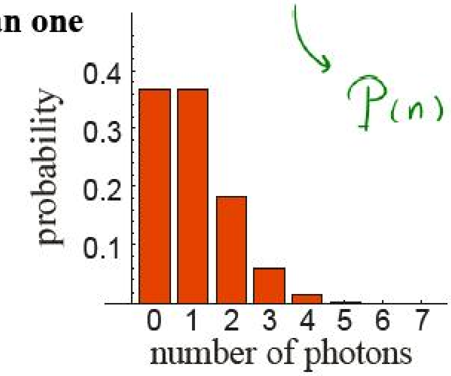
→ $n$ photons with the same $\vec{k}$ & polarization

## How to generate a photon?

- **Attenuate a laser beam?**
  - **Use a pulsed laser → attenuate to the one-photon level**



**Output will be stochastic (Poissonian statistics): sometimes zero photons, sometimes more than one**




$$P(n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

$$\lambda = \langle n \rangle$$

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$|\psi\rangle = |n_1, n_2\rangle$ a two-mode state

All insertions are taken from the following source:

# QUANTUM INFORMATION
## with light and atoms

*Lecture 1*

**Alex Lvovsky**

IISSQI-2008
Kish Island, Iran.

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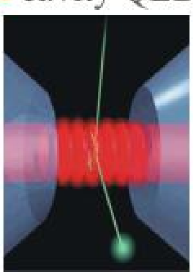
# Photon as a qubit

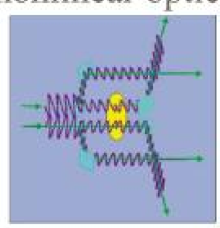
- **Among many physical media suitable for quantum computation…**

cavity QED



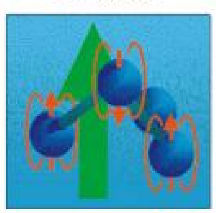
trapped ions

nonlinear optics

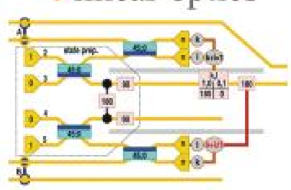


quantum dots

NMR



linear optics



- **…why study the optical one?**

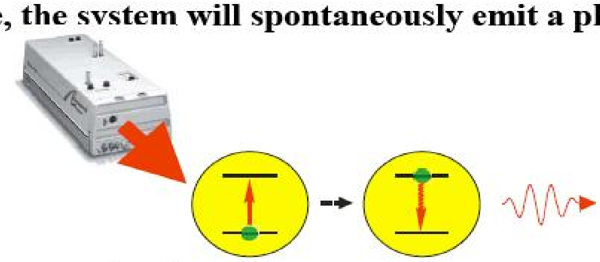
## Photon as a qubit
### (…continued)

- **Because:**
  - **A photon makes an intuitive qubit**
  - **A photon is a good carrier of quantum information**
  - **Virtually no decoherence**
  - **Efficient gate operations (Knill-Laflamme-Milburn)**
- **Challenges:**
  - **Synthesis,**
  - **Characterization**
  - **Storage**
  - **Computational gates**

  **of quantum optical states**

## How to generate a photon? (…continued)

- **Microscopic system (e.g. atom)**
  - **Excite using a laser**
  - **After a while, the system will spontaneously emit a photon**



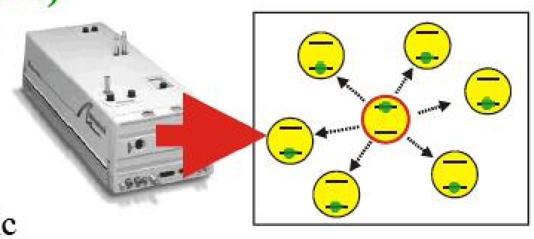
**Only one photon emitted at a time**

**System is hard to prepare and keep stable**

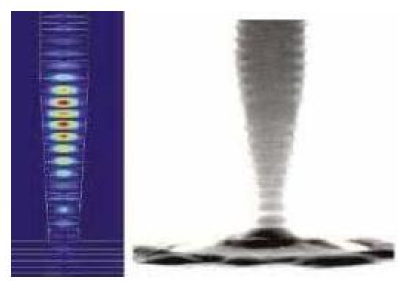
- **Nitrogen vacancies in diamond**
  - **A single structure defect in a crystal**
  - **Similar to a single atom**
  - **When excited, cannot emit more than one photon at a time**

# How to generate a photon? (…continued)

- **Mesoscopic system (e.g. a quantum dot)**
  - Microscopic elements "talk" to each other
    → One excited element
    will prevent excitation of the others
  - Only one photon emitted at a time
  - System is easier to handle than microscopic



- **Quantum dot photon sources**
  - Self-assembled
    ⇒ need to pick a good dot to work with
  - Operate at cryogenic temperatures
  - Excited electrically or optically
  - Pico-or femtosecond pulse width
  - Difficult to make transform-limited
    → verification by the Hong-Ou-Mandel dip
  - Difficult to collect
    → microcavities



[Reproduced from http://www.stanford.edu/group/yamamotogroup/]

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# Elements of nonlinear optics

- **Linear medium:**

  polarization is proportional to the EM field

$\vec{P} = \chi \vec{E}$

Isotropic

- **Nonlinear medium:**

  $P_i \propto \chi_{ij}^{(1)} E_j + \chi_{ijk}^{(2)} E_j E_k + \ldots$

  - If $E \propto e^{i\omega t}$ then $P_i \propto \ldots + e^{2i\omega t} + \ldots$

    $\rightarrow$ second harmonic generation

  - If two fields are present ($\omega_1$ and $\omega_2$) then $P_i \propto \ldots + e^{i(\omega_1+\omega_2)t} + e^{i(\omega_1-\omega_2)t} + \ldots$

    $\rightarrow$ sum, difference frequency generation

- These are classical effects
- Quantum interpretation of second harmonic generation:
  - Two photons "unite" to form a single photon of higher energy

$E = A(e^{i\omega t} + e^{-i\omega t})$

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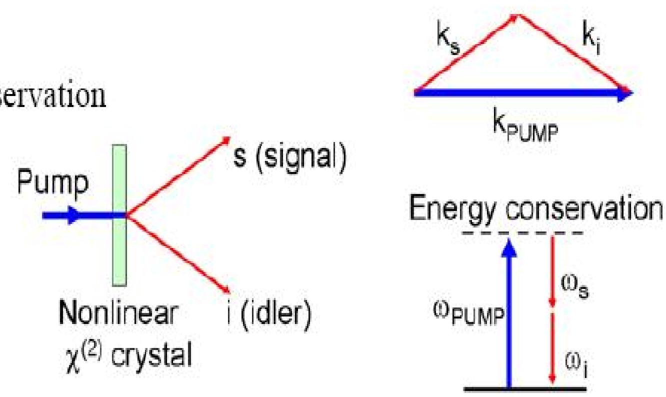
# Parametric down-conversion

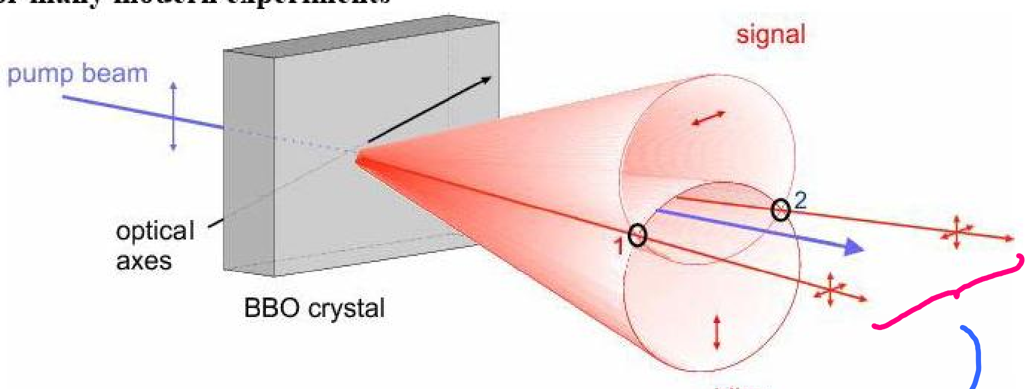
- **Quantum description**
  - Interaction energy/Hamiltonian: $H \propto \vec{E}\vec{P} \propto \sum E_i E_j E_k$
  - In the quantum form: $\hat{H} \propto ... + \hat{a}_1 \hat{a}_2^\dagger \hat{a}_3^\dagger + \text{H.c.} + ...$
  - Evolution (assume weak perturbation):
    $|\Psi(t)\rangle = e^{i\hat{H}t}|\Psi(0)\rangle \approx |0\rangle + i\hat{H}t|0\rangle = |0\rangle + ig\hat{a}_1\hat{a}_2^\dagger\hat{a}_3^\dagger t|0\rangle$
- **Interpretation:**
  - a photon of wave 1 ("pump") can split into two photons of waves 2 and 3.
  - may occur spontaneously:
    waves 2 and 3 need not be present
  - Purely quantum effect
  - Energy and momentum conservation (phase matching) must hold.
  - Main property: photons are always born in pairs.




[image by J. Lundeen from Wikipedia] $\varphi_{PUMP} = \varphi_s + \varphi_i$

## Type I and Type II down-conversion

- **Type I**
  - **Generated photons are of the same polarization**
  - **Useful for squeezing, preparation of heralded single photons, *etc.***
- **Type II**
  - **Photons have different polarizations**
  - **Emitted along two cones**
  - **Polarization-entangled biphoton** $|HV\rangle + e^{i\varphi}|VH\rangle$ **at the intersection of cones**
  - **Basis for many modern experiments**



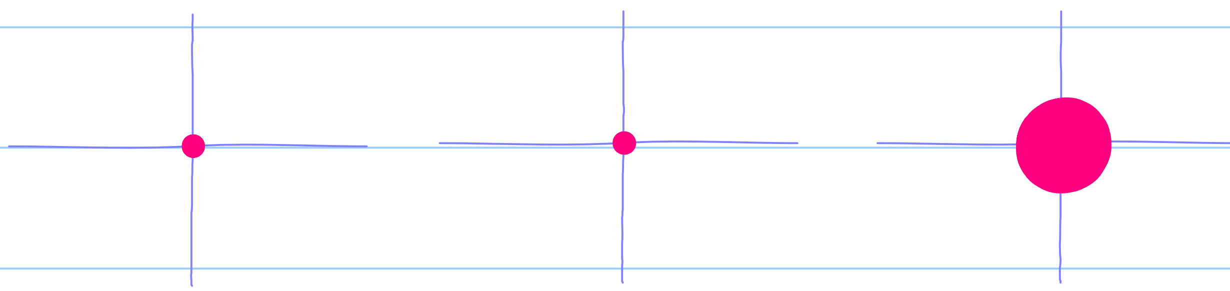

[Reproduced from http://scotty.quantum.physik.uni-muenchen.de ]

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left( |HV\rangle + e^{i\phi}|VH\rangle \right)$$

Single mode state: $|n\rangle$

$$(\Delta X)_n^2 = \langle n | X^2 | n \rangle - \langle n | X | n \rangle^2 = n + \frac{1}{2} .$$

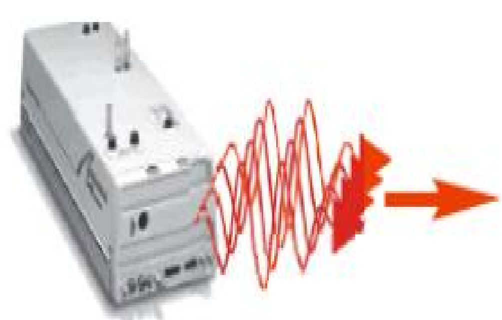
$$(\Delta P)_n^2 = \langle n | P^2 | n \rangle - \langle n | P | n \rangle = n + \frac{1}{2}$$



$|0\rangle$ $\qquad$ $|1\rangle$ $\qquad$ $|2\rangle$

Large $n$ $\longrightarrow$ Large Fluctuations in $X, P$ or in $E$ & $B$.

$|\psi\rangle = |z\rangle$ Coherent state

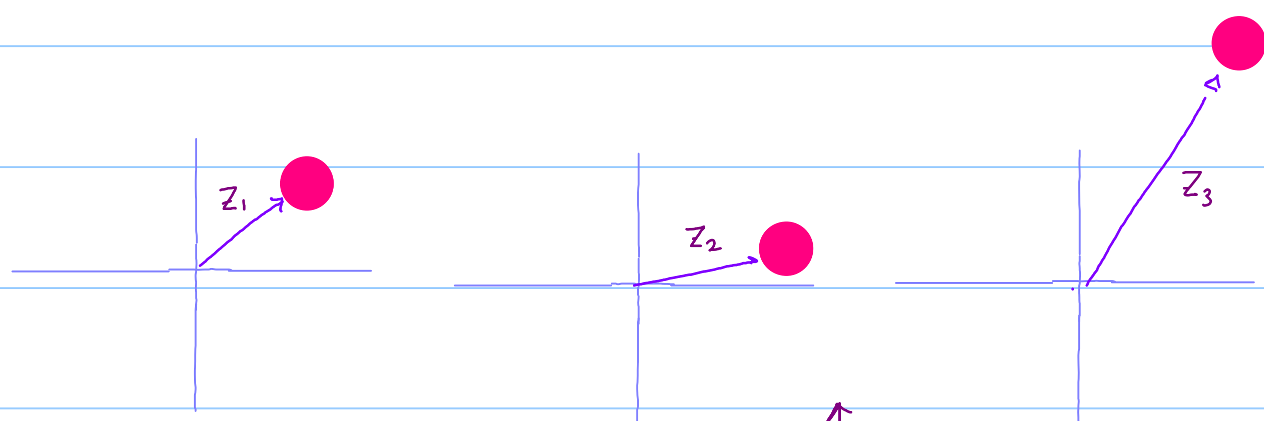


$$|z\rangle = e^{z a^\dagger} |0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

$$|z\rangle_n = e^{z a^\dagger - \bar{z} a} |0\rangle$$

$$D(z) := e^{z a^\dagger - \bar{z} a}$$

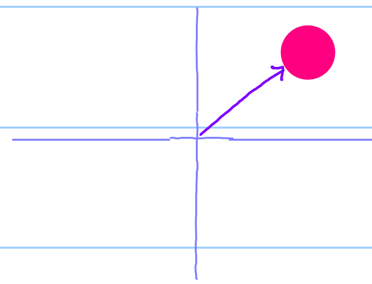
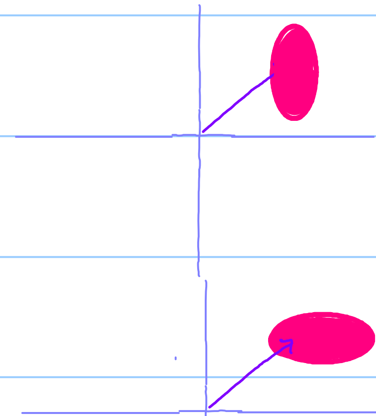
$$a|z\rangle = z|z\rangle \qquad \langle X\rangle_z = \frac{z+\bar{z}}{\sqrt{2}}$$




Coherent states

$$S(\xi) = e^{\bar{\xi} a^2 - \xi a^{\dagger 2}}$$ Squeezing Operator

$$\xi = r e^{i\phi}$$

$S(\xi)$  $\rightarrow$ 

$$S(\xi)|z\rangle = |z,\xi\rangle.$$

$$(\Delta X)^2 = \frac{1}{2} e^{-r} \qquad (\Delta p)^2 = \frac{1}{2} e^{r}$$

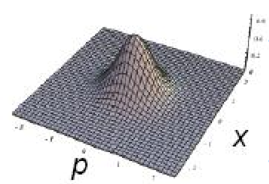
## What is squeezed light?

- **Vacuum state: light is off**
  - Quantum noise phase-independent
  - Related to shot noise in electronics
- **Squeezed vacuum state**
  - Quantum noise phase-dependent
  - At some phases, noise *below* the vacuum level
  - At other phases, excessive noise (uncertainty principle!)
- **Applications**
  - Precision interferometric meaurements (e.g. gravitation wave detection)
  - Major quantum information primitive

**Vacuum state wave function**

$$\psi_o(x) = e^{-x^2/2}$$

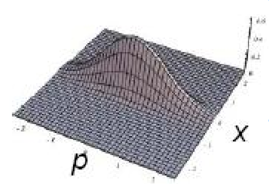
$$\psi_o(p) = e^{-p^2/2}$$




***X*-squeezed state wave function**

$$\psi_s(x) = e^{-(sx)^2/2}$$

$$\psi_s(p) = e^{-(p/s)^2/2}$$




* Problem. Normalize the above wave functions

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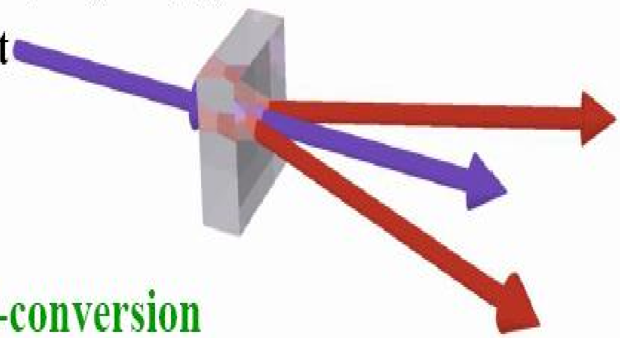
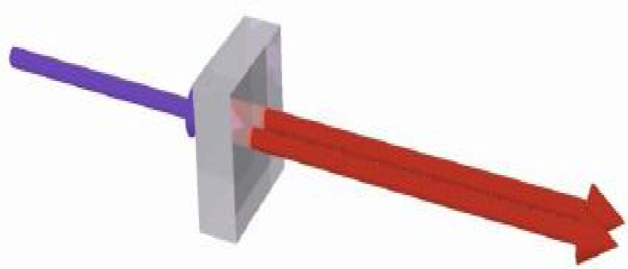
# How to produce squeezing?

- Non-degenerate parametric down-conversion
  - Photons are different in direction, frequency, polarization
  - Used e.g. to create entanglement



- Degenerate parametric down-conversion
  - Photons are identical
  - If we can generate enough pairs, output will be squeezed
  - Use optical cavity to enhance nonlinearity



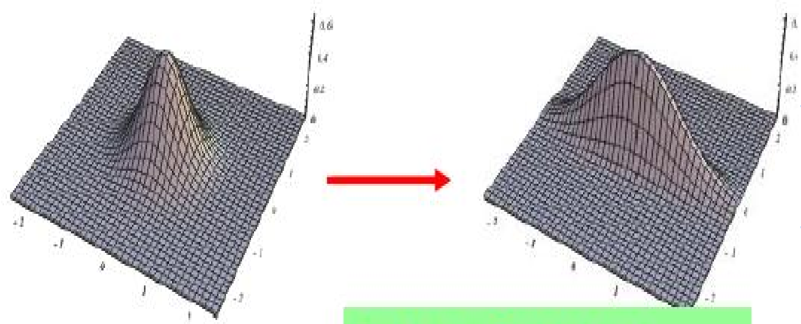
Problem. Show that the state $|0\rangle + \beta|2\rangle$ is squeezed for some values of β

# Generation of squeezed states

- **Fully degenerate down-conversion**
  - ⇒ **Generated photons are identical:** $\hat{a}_2 = \hat{a}_3$
  - ⇒ **Hamiltonian becomes** $\hat{H} \propto \hat{a}_1 \hat{a}_2^\dagger \hat{a}_3^\dagger + \text{H.c.} \;\; \rightarrow \;\; \hat{a}_1 \left(\hat{a}_2^\dagger\right)^2 + \text{H.c.}$
- **Strong pump**
  - ⇒ **Can assume classical:** $\hat{a}_1 \rightarrow i\alpha$. **Assume $\alpha$ real.**
  - ⇒ **Cannot use one-pair approximation**
- **Heisenberg evolution**
  - **For field operators:**

    $$\dot{\hat{a}}_2 = i[\hat{H}, \hat{a}_2] = 2\alpha \hat{a}_2^\dagger$$

    $$\dot{\hat{a}}_2^\dagger = 2\alpha \hat{a}_2$$



  - **For field quadratures:**

    $$\dot{\hat{X}} = 2\alpha \hat{X} \;\; \Rightarrow \;\; \hat{X}(t) = e^{2\alpha t} \hat{X}(0)$$

    $$\dot{\hat{P}} = -2\alpha \hat{P} \;\; \Rightarrow \;\; \hat{P}(t) = e^{-2\alpha t} \hat{P}(0)$$

  - ***P* shrinks, *X* expands → squeezed vacuum!**
  - **Unlike biphotons, squeezed states are "on demand"**

Problem. Repeat this calculation for a complex $\alpha$

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# MEASURING QUANTUM STATES OF LIGHT

1. By photon counting
2. By homodyne tomography

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# MEASURING THE QUANTUM STATE OF LIGHT

1. **By photon counting**
2. **By homodyne tomography**

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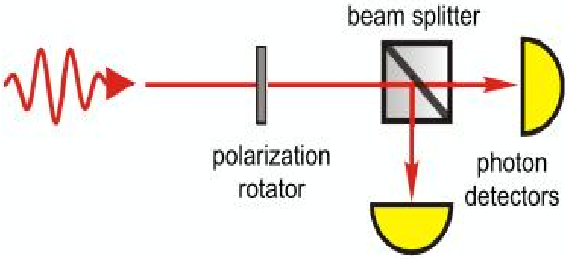
## How to characterize a quantum state?

- **A single measurement won't do**
- **Repeated, identical measurements → projection onto only one basis**
- **Need many sets of measurements in different bases (quantum tomography)**

Generally, $d^2-1$ bases are required
for full tomography of a $d$-dimensional system 1

### Example: a polarization qubit

(photon in a superposition of horizontal and vertical polarization states)




probability of horizontal polarization $=\langle H|\hat\rho|H\rangle$

$$\hat\rho=\begin{pmatrix}\rho_{HH} & \rho_{HV}\\ \rho_{VH} & \rho_{VV}\end{pmatrix}$$

probability of vertical polarization $=\langle V|\hat\rho|V\rangle$

probability of 45° polarization $=\frac{1}{2}\big(\big(\langle H|+\langle V|\big)\hat\rho\big(|H\rangle+|V\rangle\big)\big)$

prob. of right circular polarization $=\frac{1}{2}\big(\big(\langle H|-i\langle V|\big)\hat\rho\big(|H\rangle+i|V\rangle\big)\big)$

1

$$\rho = \frac{1}{2}\begin{pmatrix} 1+c & a-ib \\ a+ib & 1-c \end{pmatrix} \qquad \rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right)$$

$$\rho = \frac{1}{2}\left(I + aX + bY + cZ\right)$$

$$a = tr(X\rho) \qquad b = tr(Y\rho) \qquad c = tr(Z\rho)$$

$$a = \langle X \rangle \qquad b = \langle Y \rangle \qquad c = \langle Z \rangle$$

3 measurements are required.

$X$ اندازه‌گیری $\longrightarrow a$

$Y$ " $\longrightarrow b$

$Z$ " $\longrightarrow c$

Tomography معمولاً تعداد اندازه‌گیری‌ها محدود است $(N_i)$

در حالت کلی:

$$\rho = \frac{1}{d}\left(I + \sum_{i=0}^{d^2-1} r_i T_i\right)$$

$$Tr(T_i T_j) = d\,\delta_{ij} \rightarrow r_i = Tr(\rho T_i)$$

اندازه‌گیری در $d^2-1$ پایه مورد نیاز است.

## Quantum measurement of the Bell state

d state $|\Psi^-\rangle = |HV\rangle - |VH\rangle$

o observes $V$

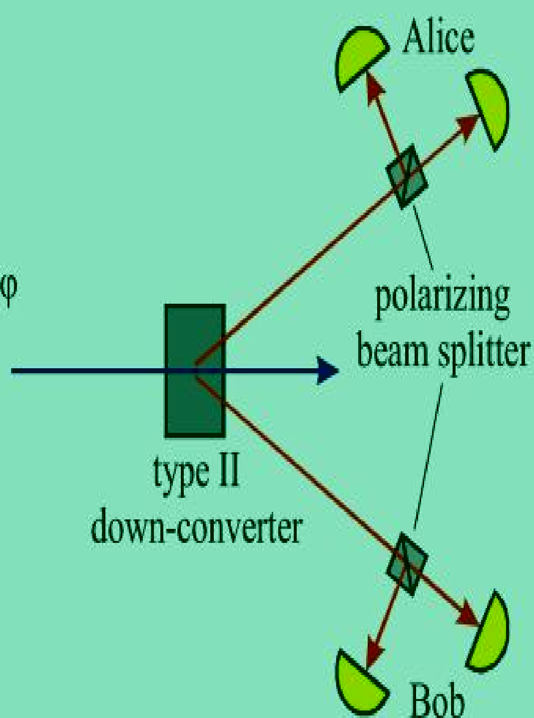
o observes $H$

**ufficient.**

$VH\rangle$ with any φ

ure

**larizers 45°**




$|V'H'\rangle$ remains the same

$-|V'V'\rangle$ changes

**tate is indeed** $|\Psi^-\rangle$

Problem. Verify this.

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## of the Bell state

- **Measuring an entangle**
  - **Perfect anticorrelation:**
    - If Alice observes $H$, Bol
    - If Alice observes $V$, Bol
  - **This measurement is ins**

    The state can be $|HV\rangle + e^{i\varphi}|$

    or even an unentangled mixt

    $|HV\rangle\langle HV| + |VH\rangle\langle VH|$
  - **To determine φ, turn po**

    $|H\rangle \rightarrow (|H'\rangle + |V'\rangle)/\sqrt{2}$

    $|V\rangle \rightarrow (|H'\rangle - |V'\rangle)/\sqrt{2}$

    Then

1

    $|HV\rangle - |VH\rangle \rightarrow |H'V'\rangle -$

    $|HV\rangle + |VH\rangle \rightarrow |H'H'\rangle -$

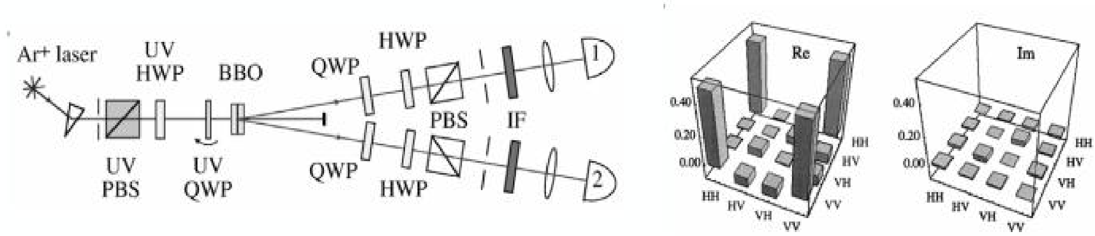
    $\Rightarrow$ **We can verify that the s**

# Quantum tomography by photon counting

- **Example:**

  A. G. White *et al.*, PRL 83, 3103 (1999)

  - **Tomography of a two-mode, partially entangled state**




- **Measurements complete. What next?**
  - **Need to determine the density matrix from measurement results**
  - **Likelihood function**

$$\mathfrak{L}(\hat{\rho}) = \prod_{\text{measurements}} \mathrm{pr}_i(\hat{\rho}) \quad 2$$

(where $i$ is the number of the measurement, $\rho$ is the density matrix)

  - **Likelihood-maximization algorithm**

Finds, among all possible density matrices, the one that maximized $\mathfrak{L}$

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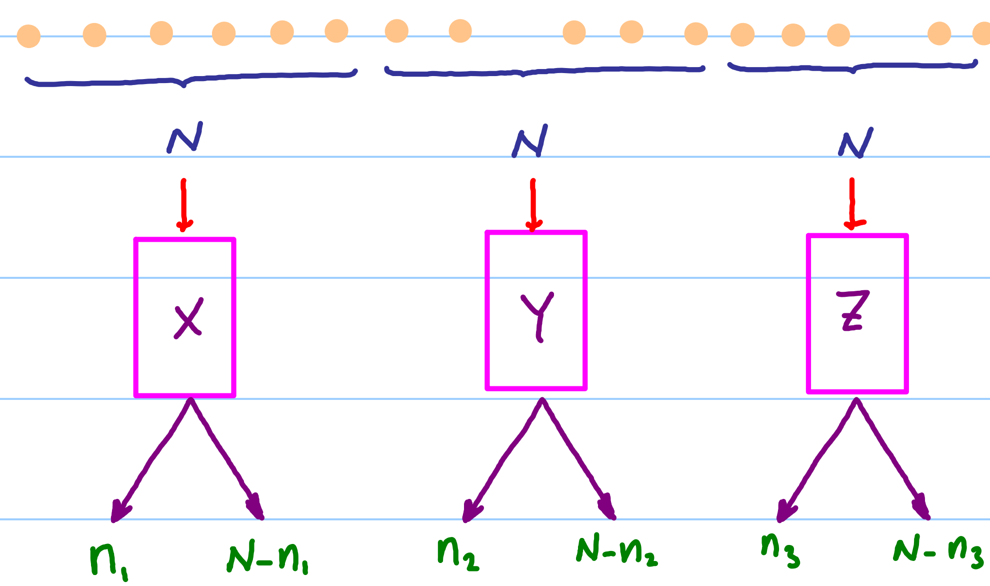


$$\rho = \frac{1}{2}\begin{pmatrix} 1+c & a-ib \\ a+ib & 1-c \end{pmatrix}$$

بهترین تخمین برای $\rho$ چیست؟

$$Pr(+|X) = \langle x+|\rho|x+\rangle = \frac{1+a}{2}$$

$$Pr(+|Y) = \langle y+|\rho|y+\rangle = \frac{1+b}{2}$$

$$Pr(+|Z) = \langle z+|\rho|z+\rangle = \frac{1+c}{2}$$

$$Pr_X(n_1, N-n_1) = \binom{N}{n_1}\left(\frac{1+a}{2}\right)^{n_1}\left(\frac{1-a}{2}\right)^{N-n_1}$$

با عبارات مشابه برای $Y$ , $Z$ . ←

$$\mathcal{L}(\hat{\rho}) \equiv Pr_X \, Pr_Y \, Pr_Z = \mathcal{L}_{N, n_1, n_2, n_3}(a, b, c).$$

Maximize $\mathcal{L}(a,b,c)$ with respect to $a, b, c \longrightarrow \rho$.

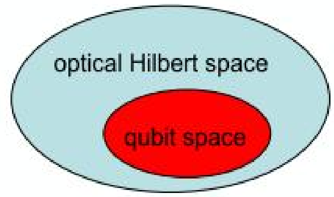
# Tomography by photon counting

## *Drawbacks*

- **Polarization qubit**

$$\alpha|H\rangle+\beta|V\rangle=\alpha|1_H,0_V\rangle+\beta|0_H,1_V\rangle\ldots$$

$$\boxed{+\gamma|0_H,0_V\rangle+\delta|1_H,1_V\rangle+\varepsilon|2_H,3_V\rangle\ldots}\ ?$$




- **Traditional approach neglects non-qubit terms**
  - $\rightarrow$ incomplete state characterization
  - $\rightarrow$ incorrect evaluation of experimental quantum algorithms
  - $\rightarrow$ postselection $\Rightarrow$ loss of scalability
- **New technology: number discriminating detector**
  - "Regular" photon detector: "click" or "no click"
  - Number discriminating detector: can determine the number **of photons**
  - Still, no phase information

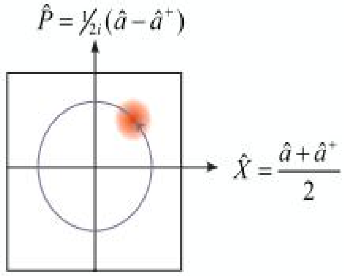
Problem. suppose you have many highly-efficient "regular" detectors. Can you use them to construct a discriminating detector?

# MEASURING THE QUANTUM STATE OF LIGHT

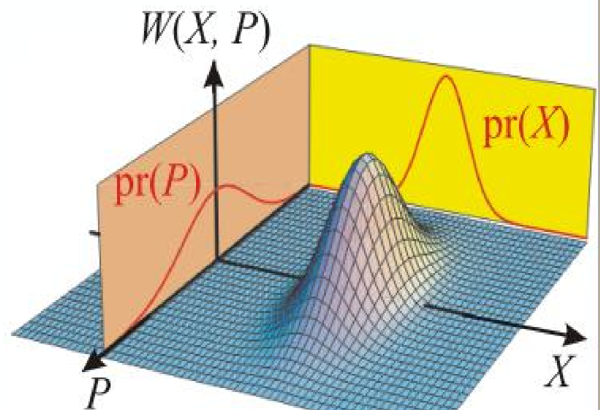
1. By photon counting
2. **By homodyne tomography**

# Wigner Function

- **Quantum mechanics → Uncertainty principle**
  → phase space probability density cannot be defined
  → only individual quadratures can be measured
- **Phase-space "quasi"probability density (Wigner function)**
  → projection onto each quadrature determines its probability density




$$W(X,P) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(ipq)\left\langle X - \frac{q}{2}\right|\hat{\rho}\left|X + \frac{q}{2}\right\rangle dq$$

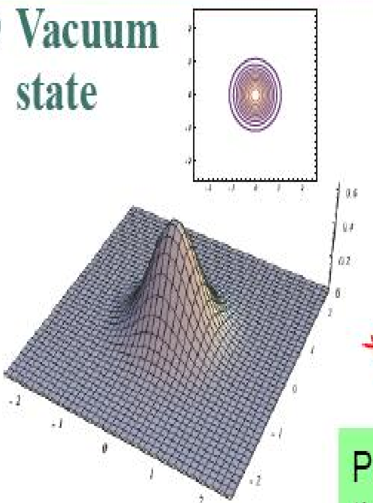



- **Properties**
  → completely describes a quantum state
  → real, normalized
  → not necessarily positive definite

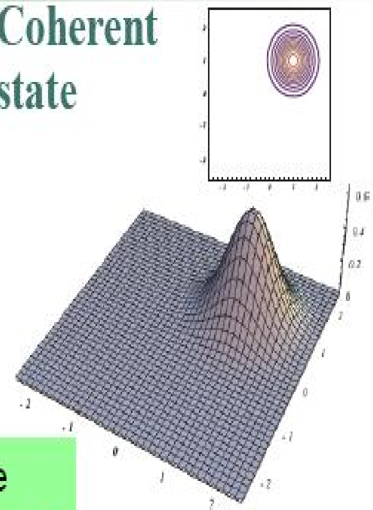
E.P. Wigner, Phys. Rev. **40**, 749 (1932)

# Examples of Wigner functions

- Vacuum state

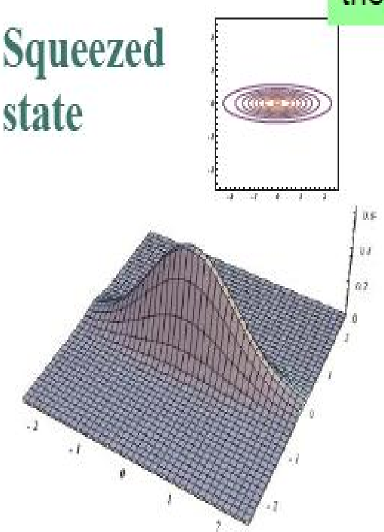


- Coherent state

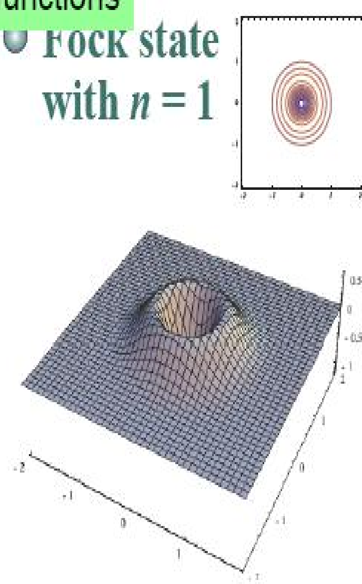


Problem. Calculate these Wigner functions

- Squeezed state



- Fock state with $n = 1$



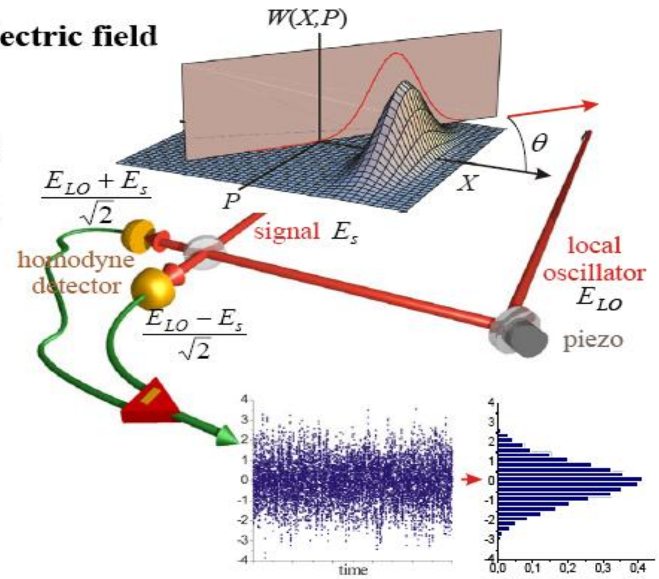
# Homodyne tomography

- **Phase-sensitive measurements of electric field**
  - $\rightarrow$ cannot be done directly
  - $\rightarrow$ use interference with local oscillator
- **Measure subtraction photocurrent**

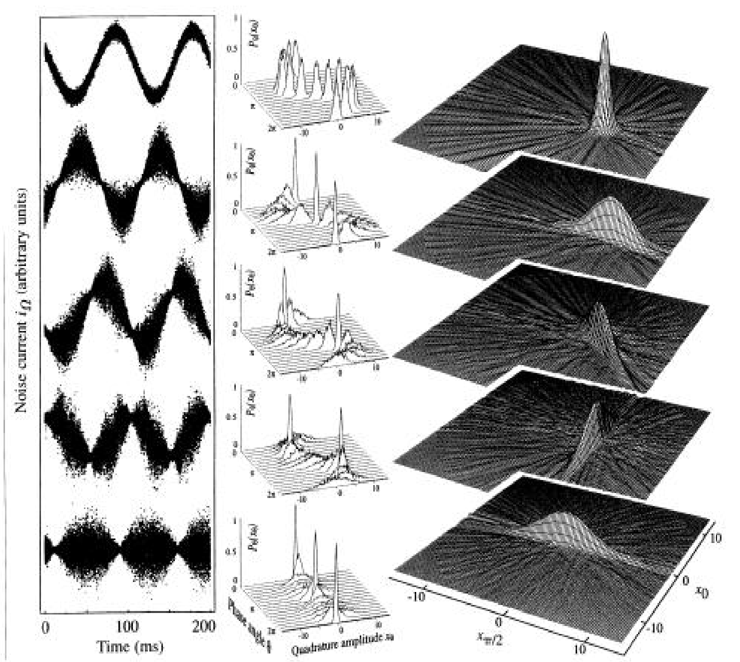
$$I_- = \left|\frac{E_{LO}+E_s}{\sqrt{2}}\right|^2 - \left|\frac{E_{LO}-E_s}{\sqrt{2}}\right|^2 = 2E_{LO}E_s$$

  Assume $|E_{LO}| >> |E_s|$ so the local oscillator can be treated classically

  $\Rightarrow$ Subtraction photocurrent $\propto$ signal field ($= X_\theta$)
- **Many measurements** **$\rightarrow$ histogram pr($X_\theta$)** ("marginal distribution")
- **Set of pr($X_\theta$) for all $\theta$**
  - $\rightarrow$ Wigner function W($X$, $P$) (via inverse Radon transform)
  - $\rightarrow$ Density matrix $\hat{\rho}$ (via likelihood maximization)




# Example 1: squeezed states

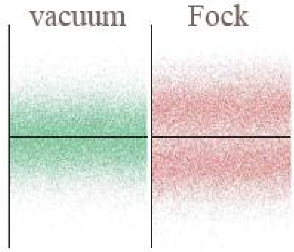



noise traces | quadrature distributions | reconstructed Wigner functions

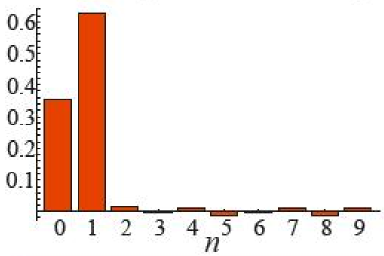
[G. Breitenbach, S. Schiller & J. Mlynek, Nature **387**, 471 (1997)]

# Example 2:
# Single-photon Fock state tomography

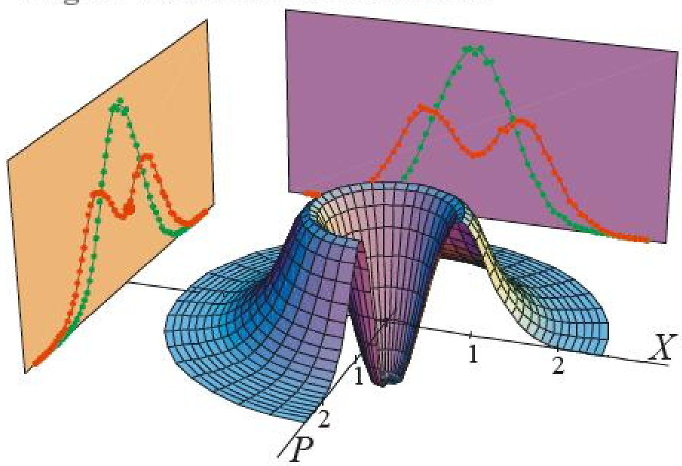
- Quadrature noise: raw data, 45000 pts




- Density matrix (diagonal elements)




- Wigner function reconstruction




Efficiency: 62%

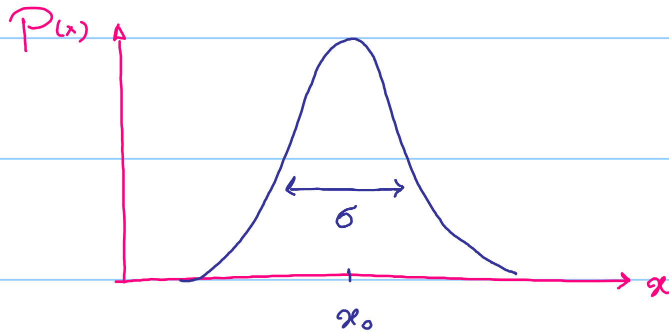
Wigner function is negative in the origin of the phase space

A. I. Lvovsky et al., PRL **87**, 050402 (2001)

Institute for Quantum Information Science at the University of Calgary

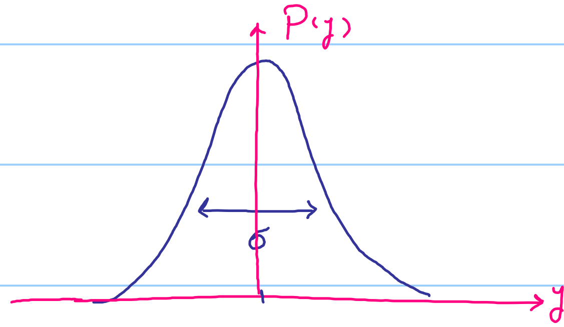
# Gaussian States.

$$P(x) = A\, e^{-\frac{1}{2\sigma^2}(x-x_0)^2}$$




$\langle X \rangle = x_0$

$\langle X^2 \rangle - \langle X \rangle^2 = \sigma^2$

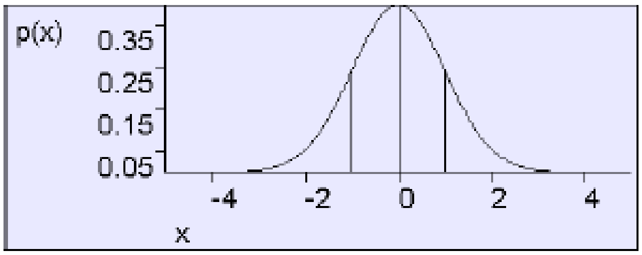



$\langle Y \rangle = 0$

$\langle Y^2 \rangle = \sigma^2$

# Unit variance Gaussian

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$




$$E[X] = 0$$

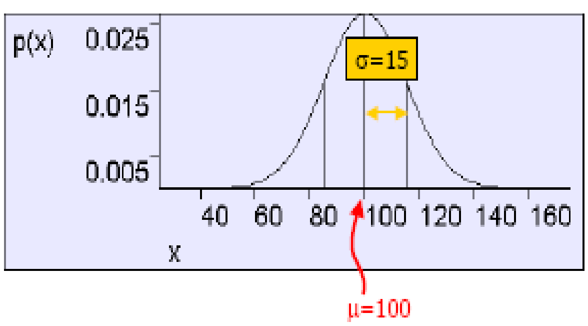
$$\mathrm{Var}[X] = 1$$

$$H[X] = -\int_{x=-\infty}^{\infty} p(x) \log p(x) dx = 1.4189$$




# General Gaussian

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$




$$E[X] = \mu$$

$$\mathrm{Var}[X] = \sigma^2$$

# Bivariate Gaussians

Write r.v. $\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix}$ Then define $X \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ to mean

$$p(\mathbf{x}) = \frac{1}{2\pi \, \| \boldsymbol{\Sigma} \|^{1/2}} \exp\left( -\tfrac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right)$$

Where the Gaussian's parameters are...

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma^2{}_x & \sigma_{xy} \\ \sigma_{xy} & \sigma^2{}_y \end{pmatrix}$$

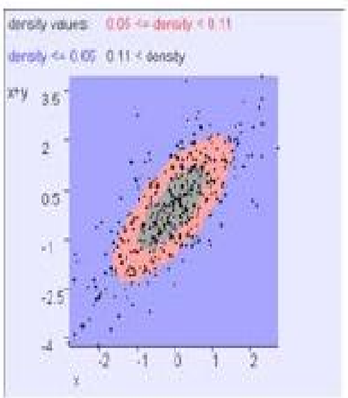
Where we insist that $\Sigma$ is symmetric non-negative definite

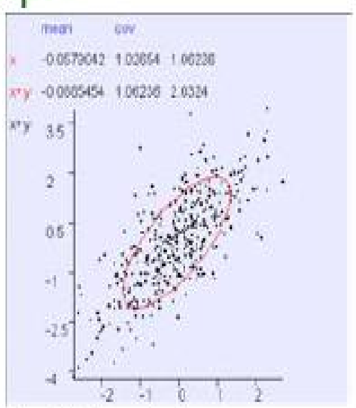
It turns out that E[X] = $\mu$ and Cov[X] = $\Sigma$. (Note that this is a resulting property of Gaussians, not a definition)*

*This note rates 7.4 on the pedanticness scale



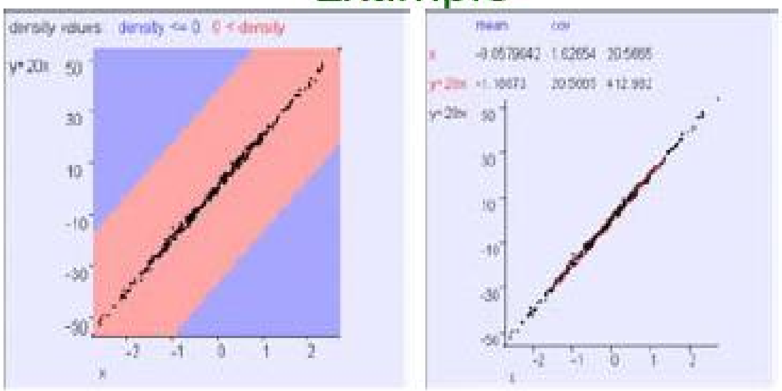

# Example




In this example, x and "x+y" are clearly not independent

# Example




In this example, x and "20x+y" are clearly not independent




# Multivariate Gaussians

Write r.v. $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix}$ Then define $X \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ to mean

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{m/2} \, \| \boldsymbol{\Sigma} \|^{1/2}} \exp\left( -\tfrac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \, \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right)$$

Where the Gaussian's parameters have...

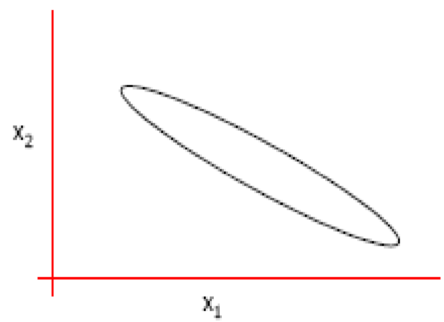
$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma^2{}_1 & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{12} & \sigma^2{}_2 & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1m} & \sigma_{2m} & \cdots & \sigma^2{}_m \end{pmatrix}$$

Where we insist that $\Sigma$ is symmetric non-negative definite

Again, $E[X] = \mu$ and $Cov[X] = \Sigma$. (Note that this is a resulting property of Gaussians, not a definition)

# General Gaussians

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma^2{}_1 & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{12} & \sigma^2{}_2 & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1m} & \sigma_{2m} & \cdots & \sigma^2{}_m \end{pmatrix}$$



# Phase Space

$(R_1, \ldots, R_{2N}) = (X_1, P_1, \ldots, X_N, P_N)$, the *canonical commutation re*
pressed as

$$[R_k, R_l] = i\sigma_{k,l}\mathbb{I},$$

where the skew-symmetric $2N \times 2N$-matrix $\sigma$ is given by

$$\sigma = \bigoplus_{i=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$D_\xi := e^{i\xi^T \sigma \hat{R}}$$

$$\xi = (\xi_1, \xi_2, \ldots \xi_{2N}) = (x_1, p_1, x_2, p_2 \cdots x_N, p_N)$$

$N=1$ $$D_\xi = e^{i(\xi_1\ \xi_2)\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix}\hat{R}_1 \\ \hat{R}_2\end{pmatrix}}$$

$$= e^{i(\xi_1\hat{R}_2 - \xi_2\hat{R}_1)} = e^{i(x\hat{P} - p\hat{X})}$$

$$\hat{X} = \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}} \qquad \hat{P} = \frac{\hat{a}-\hat{a}^\dagger}{i\sqrt{2}}$$

$$i(x\hat{P} - p\hat{X}) = x\left(\frac{a-a^\dagger}{\sqrt{2}}\right) - ip\left(\frac{a+a^\dagger}{\sqrt{2}}\right)$$

$$= \left(\frac{x-ip}{\sqrt{2}}\right)a - \left(\frac{x+ip}{\sqrt{2}}\right)a^\dagger = \bar{z}a - za^\dagger$$

$$\boxed{\begin{aligned} D_\xi &= e^{i\xi^T\sigma\hat{R}} \\ D_{x,p} &= e^{i(x\hat{P}-p\hat{X})} \\ D_{z,\bar{z}} &= e^{\bar{z}a - za^\dagger}\end{aligned}}$$

$$D_\xi D_\eta = e^{-i\xi^T\sigma\eta} D_{\xi+\eta}$$

Characteristic Function $\chi_\rho(\xi) = \mathrm{tr}(D_\xi \rho)$

$$\rho = \frac{1}{(2\pi)^N} \int (d^{2N}\xi)\ \chi_\rho(\xi)\ D_\xi^\dagger$$

Wigner function

$$W_\rho(\xi) := \frac{1}{(2\pi)^N} \int (d^{2N}\eta)\ e^{i\xi^T \sigma \eta}\ \chi_\rho(\eta)$$

$$\langle X \rangle = \int dx\,dp\ W(x,p)\ x$$

$$\langle P \rangle = \int dx\,dp\ W(x,p)\ p$$

$$\langle f(x,p) \rangle = \int dx\,dp\ W(x,p)\ f(x,p).$$

$$\mathrm{tr}(A_1 A_2) = (2\pi)^N \int d^{2N}\xi \; W_{A_1}(\xi) W_{A_2}(\xi)$$

averages: $d_k := \langle R_k \rangle$

Correlations: $\gamma_{kl} := \langle \{ R_k - d_k , R_l - d_l \} \rangle$

uncertainty principle + $\rho \geq 0$

$$\Downarrow$$

$$\gamma + i\sigma \geq 0$$

Gaussian States:

$$\chi_\rho(\xi) = e^{i\xi^T \sigma d - \frac{1}{4} \xi^T \sigma^T \gamma \sigma \xi}$$

# Examples:

Coherent State: $|z\rangle\langle z|$ $\quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Squeezed State: $|\Omega\rangle_s\langle\Omega|$ $\quad \gamma = \begin{pmatrix} \frac{1}{d} & \\ & d \end{pmatrix}$

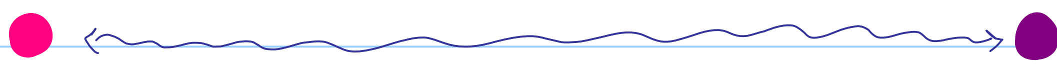
Thermal State: $\frac{1}{Z}e^{-\beta H} = \sum_{n=0}^{\infty} \frac{1}{Z} e^{-\beta(n+\frac{1}{2})} |n\rangle\langle n|$

$\bar{n} = \frac{1}{e^{\beta}-1}$ $\quad \gamma = (2\bar{n}+1)$

# Seperability of Gaussian States

## Bi-partite

1) 2-modes.



## Peres-Horodecki separability criterion for continuous variable systems*

R. Simon
*The Institute of Mathematical Sciences, Tharamani, Chennai 600 113, India*

(February 1, 2008)

The Peres-Horodecki criterion of positivity under partial transpose is studied in the context of separability of bipartite continuous variable states. The partial transpose operation admits, in the continuous case, a geometric interpretation as mirror reflection in phase space. This recognition leads to uncertainty principles, stronger than the traditional ones, to be obeyed by all separable states. For all bipartite Gaussian states, the Peres-Horodecki criterion turns out to be necessary and sufficient condition for separability.

PRL (1999)

$$\hat{\rho} \longrightarrow \hat{\rho}^T \iff W(q,p) \longrightarrow W(q,-p).$$

$$W(q,p) = \pi^{-2} \int d^2 q' \langle q - q' | \hat{\rho} | q + q' \rangle \exp(2i\, q' \cdot p).$$

$$PT: \quad W(q_1, p_1, q_2, p_2) \longrightarrow W(q_1, p_1, q_2, -p_2).$$

$$PT: \quad \xi \longrightarrow \Lambda\xi, \quad \Lambda = \mathrm{diag}(1,1,1,-1).$$

$\rho$ : $\quad \gamma + \frac{i}{2}\sigma \geq 0$

$\rho^T$ : $\quad \gamma \longrightarrow \tilde{\gamma} + \frac{i}{2}\sigma \geq 0$

Seperable? $\rightarrow \quad \tilde{\gamma} + \frac{i}{2}\sigma \geq 0$

For Gaussian states : $$V_0 = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix}.$$

$$4(ab - c_1^2)(ab - c_2^2) \geq (a^2 + b^2) + 2|c_1 c_2| - 1/4.$$

# Inseparability criterion for continuous variable systems

Lu-Ming Duan[1,2*], G. Giedke[1], J. I. Cirac[1], and P. Zoller[1]

[1]*Institut für Theoretische Physik, Universität Innsbruck, A-6020 Innsbruck, Austria*

[2]*Laboratory of Quantum Communication and Quantum Computation, University of Science and Technology of China, Hefei 230026, China*

An inseparability criterion based on the total variance of a pair of Einstein-Podolsky-Rosen type operators is proposed for continuous variable systems. The criterion provides a sufficient condition for entanglement of any two-party continuous variable states. Furthermore, for all the Gaussian states, this criterion turns out to be a necessary and sufficient condition for inseparability.

**PACS numbers:** 03.67.-a, 42.50.Dv, 89.70.+c

PRL 2000

$$\psi(x_1, x_2) = \frac{1}{2\pi}\int dp\, e^{ip(x_1 - x_2 - x_0)} = \delta(x_1 - x_2 - x_0)$$

$$\tilde{\psi}(p_1, p_2) = \frac{1}{2\pi}\int e^{ip_1 x_1 + ip_2 x_2}\, \psi(x_1, x_2)$$

$$= \frac{1}{2\pi}\int e^{ip_1 x_1 + ip_2 x_2}\, \delta(x_1 - x_2 - x_0)$$

$$= \frac{1}{2\pi}\int e^{ip_1(x_2 + x_0) + ip_2 x_2}\, dx_2 \equiv \delta(p_1 + p_2)$$

$$\hat{u} = |a|\,\hat{x}_1 + \frac{1}{a}\,\hat{x}_2$$

$$\hat{v} = |a|\,\hat{p}_1 - \frac{1}{a}\,\hat{p}_2$$

$\rho$ is separable : $\Rightarrow$ $\langle (\Delta\hat{u})^2 + (\Delta\hat{v})^2 \rangle \geq a^2 + \frac{1}{a^2}$

$$M_s^{II} = \begin{pmatrix} n_1 & & c_1 & \\ & n_2 & & c_2 \\ c_1 & & m_1 & \\ & c_2 & & m_2 \end{pmatrix},$$

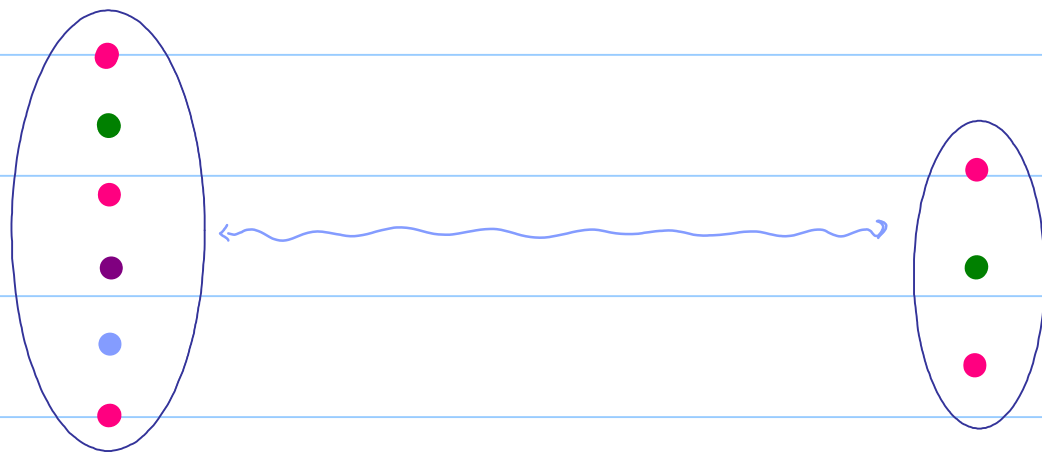
Theorem 2 (necessary and sufficient inseparability criterion for Gaussian states): A Gaussian state $\rho_G$ is separable if and only if when expressed in its standard form II, the inequality (3) is satisfied by the following two EPR-type operators

$$\hat{u} = a_0\hat{x}_1 - \frac{c_1}{|c_1|}\frac{1}{a_0}\hat{x}_2, \tag{15a}$$

$$\hat{v} = a_0\hat{p}_1 - \frac{c_2}{|c_2|}\frac{1}{a_0}\hat{p}_2, \tag{15b}$$

where $a_0^2 = \sqrt{\frac{m_1-1}{n_1-1}} = \sqrt{\frac{m_2-1}{n_2-1}}$.

bi-partite $n$-modes:



# Separability Criterion for all bipartite Gaussian States


G. Giedke(1), B. Kraus(1), M. Lewenstein(2), and J. I. Cirac(1)

*(1) Institut für Theoretische Physik, Universität Innsbruck, A-6020 Innsbruck, Austria*

*(2) Institut für Theoretische Physik, Universität Hannover, 30163 Hannover, Germany*



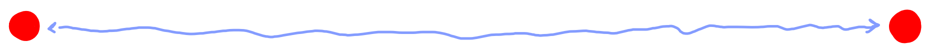
(Dated: February 1, 2008)



We provide a necessary and sufficient condition for separability of Gaussian states of bipartite systems of arbitrarily many modes. The condition provides an operational criterion since it can be checked by simple computation. Moreover, it allows us to find a pure product–state decomposition of any given separable Gaussian state. Our criterion is independent of the one based on partial transposition, and is strictly stronger.


2001

# Entanglement of formation (EoF)



## Entanglement of formation for symmetric Gaussian states

G. Giedke[1,3], M.M. Wolf[1,2], O. Krüger[2], R.F. Werner[2], and J. I. Cirac[1]

*(1) Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, Garching, D-85748, Germany.*
*(2) Institut für Mathematische Physik, Mendelsohnstr. 3, D-38106 Braunschweig, Germany*
*(3) Institut für Quantenelektronik, ETH Zürich, Wolfgang-Pauli-Straße, CH-8093 Zürich, Switzerland*

(Dated: February 1, 2008)

We show that for a fixed amount of entanglement, two-mode squeezed states are those that maximize Einstein–Podolsky–Rosen–like correlations. We use this fact to determine the entanglement of formation for all symmetric Gaussian states corresponding to two modes. This is the first instance in which this measure has been determined for genuine continuous variable systems. 2003 PRL

$$\gamma = \begin{pmatrix} n & 0 & k_x & 0 \\ 0 & n & 0 & -k_p \\ k_x & 0 & m & 0 \\ 0 & -k_p & 0 & m \end{pmatrix}. \qquad (13)$$

We will concentrate here on symmetric states, i.e. those which are invariant under exchange of subindices A and B and therefore fulfilling $m = n$. Without loss of generality we can choose $k_x \geq k_p \geq 0$. In this case, $\gamma$ is a CM iff $n^2 - k_x^2 \geq 1$ and describes an entangled state iff $1 > (n - k_x)(n - k_p)$ [13]. Next we apply local (unitary)

$$\Delta(\sigma) = \sqrt{(n - k_x)(n - k_p)} =: \delta$$

$$f(\Delta) = c_+(\Delta)\log[c_+(\Delta)] - c_-(\Delta)\log[c_-(\Delta)], \qquad (17)$$

$$C_{\pm} = \frac{\left(\sqrt{\Delta} \pm \sqrt{\Delta^{-1}}\right)^2}{4}$$

$$EOF = f(\Delta)$$

---

# Gaussian Unitaries.

$$\rho \longmapsto U\rho U^\dagger, \qquad U = \exp\left(\frac{i}{2}\sum_{k,l} H_{k,l} R_k R_l\right),$$

$$S := e^{H\sigma} \qquad S \in Sp(2N, R)$$

symplectic

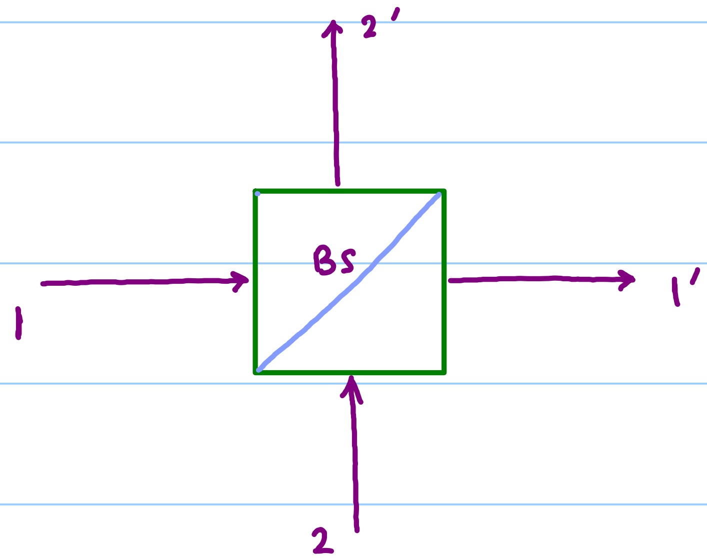
$$S \sigma S^T = \sigma$$

$$\gamma \longrightarrow \gamma' = S \gamma S^T$$

$$S_{\mathrm{BS}} = \begin{pmatrix} \sqrt{t}\mathbb{I}_2 & \sqrt{1-t}\mathbb{I}_2 \\ -\sqrt{1-t}\mathbb{I}_2 & \sqrt{t}\mathbb{I}_2 \end{pmatrix}, \qquad t \in [0,1],$$

Passive Transformation

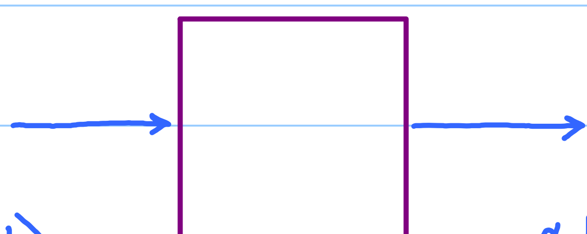
Beam Splitter




انرژی و تعداد فوتون‌ها

ثابت می‌ماند.

$$S_{\mathrm{PS}} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}, \qquad \phi \in [0, 2\pi).$$



$\alpha|0\rangle + \beta|1\rangle$ $\quad$ $\alpha|0\rangle + \beta e^{i\phi}|1\rangle$

$$U = e^{\frac{x}{2}(a^2 - a^{\dagger 2})}$$

Active Transformations

Squeezing operator

$$S = \begin{pmatrix} e^{-x} & \\ & e^{x} \end{pmatrix}$$

as follows: For any covariance matrix $\gamma$ of a system with $N$ degrees of freedom, there exists an $S \in Sp(2N, \mathbb{R})$ such that

$$S\gamma S^T = \bigoplus_{i=1}^{N} s_i \mathbb{I}_2. \tag{3.3}$$

The numbers $s_1, \ldots, s_N$ can be identified to be given by the positive part of the spectrum of $i\sigma\gamma$. This is the *normal mode decomposition*, resulting from the familiar procedure of decoupling a coupled system of harmonic oscillators. The covariance matrix of Eq. (3.3) is

Gaussian Channels.

$$\rho' = tr_E\left(U(\rho \otimes \rho_E)U^\dagger\right)$$

$$\gamma \longrightarrow \gamma' = F^T \gamma F + G$$

$$W_\xi \longrightarrow W'_\xi = W_{F\xi}\, e^{-\frac{1}{2}\xi^T G \xi}$$